Dear student following is a Moderate level [O ● O] test paper. Score of 18 Marks in 15 Minutes would be a satisfactory performance. Questions 1-10(+3,–1). (All questions have only one option correct).

Q.1 Use the function $f(x) = x^{1/x}$, $x > 0$ to determine the bigger of the two numbers e^{π} and π^{e}

(A) $e^{\pi} \le \pi^{e}$ (B) $e^{\pi} > \pi^{e}$

(C) $e^{\pi} \ge \pi^{e}$ (D) $e^{\pi} < \pi^{e}$

Q.2 If $f''(x) + f'(x) + f^2(x) = x^2$ be the differential equation of a curve and let P be the point of maxima then number of tangents which can be drawn from P to $x^2 - y^2 = a^2$ is are :

(A) 2 (B) 1

(C) 0 (D) Either 1 or 2

Q.3 The set of all values of 'b' for which the function $f(x) = (b^2 - 3b + 2)(\cos^2 x - \sin^2 x) + (b - 1)x + \sin 2$ does not possesses stationary point is :

(A) $[1, \infty)$ (B) $(0, 1) \cup (1, 4)$

(C) $\frac{3}{2}, \frac{5}{2}$ (D) None of these

Q.4 If $f''(x) > 0$ $f'(1) = 0$ such that $g(x) = f(\cot^2 x + 2\cot x + 2)$, where $0 < x < \pi$ then the interval in which g(x) is decreasing is :

(A) $(0, \pi)$ (B) $\left(\frac{\pi}{2}, \pi\right)$

(C) $\left(\frac{3\pi}{2}, \pi\right)$ (D) $\left(0, \frac{3\pi}{2}\right)$

Q.5 The equation $\sin x + x = 0$ has at least one root in interval

(A) $\left(-\frac{\pi}{2}, 0\right)$ (B) $(0, \pi)$

(C) $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ (D) None of these

Q.6 A ball is dropped from platform 19.6 m high. Its position function is

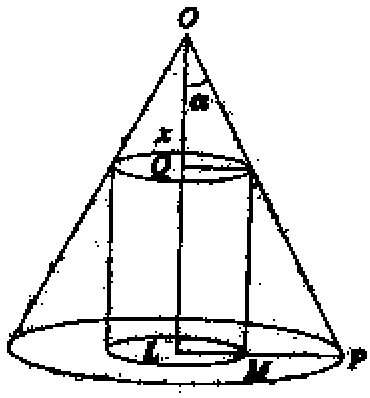
(A) $x = -4.9t^2 + 19.6 \quad (0 \le t \le 1)$

(B) $x = -4.9t^2 + 19.6 \quad (0 \le t \le 2)$

(C) $x = -9.8t^2 + 19.6 \quad (0 \le t \le 2)$

(D) $x = -4.9t + 19.6 \quad (0 \le t \le 2)$

Q.7 A given right circular cone has a volume p, and the largest right circular cylinder that can be inscribed in the cone has a volume q. Then p : q is



(A) 9 : 4

(B) 8 : 3

(C) 7 : 2 (D) None of these

Q.8 Three normals are drawn to the parabola $y^2 = 4x$ from the point (c, 0). These normals are real and distinct when

(A) c = 0 (B) c = 1

(C) c = 2 (D) c = 3

Q.9 The set of values of a for which $f(x) = (a^2 - 3a + 2)(\cos^2(x/4) - \sin^2(x/4)) + (a - 1)x + \sin 1$ doesn't possess critical points is

(A) $[1, \infty)$ (B) $(-2, 4)$

(C) $(1, 3) \cup (3, 5)$ (D) $(0, 1) \cup (1, 4)$

Q.10 For a > 0, the value of a for which the equation $ax^2 = \log x$ possess a single root is

(A) 1/2 (B) 1/2e

(C) 1/e (D) $2e^{-1}$

MATHEMATICS IIT JEE (JULY 4th WEEK CLASS TEST 3) (DERIVATE & IT'S APP.) ANSWER KEY

Name : .... Roll No. : ....

	A	B	C	D		A	B	C	D		A	B	C	D
1	○	○	○	○	4	○	○	○	○	7	○	○	○	○
2	○	○	○	○	5	○	○	○	○	8	○	○	○	○
3	○	○	○	○	6	○	○	○	○	9	○	○	○	○
										10	○	○	○	○

ANSWER KEY

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	B	A	C	D	B	B	A	D	D	B

SOLUTIONS

Sol.1 (B)

$f(x) = x^{1/x}, x > 0$

let $y = f(x) = x^{1/x}$

Taking log on both side we have

$$\log y = \frac{1}{x}\log x$$

Differentiating both side we have

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{x^2} + \log x\left(-\frac{1}{x^2}\right)$$

or $$\frac{dy}{dx} = y\left[\frac{1-\log x}{x^2}\right]$$

$$\therefore \quad f'(x) = \frac{x^{1/x}}{x^2}[1 - \log x]$$

let $f'(x) = 0$;

$\Rightarrow \log x = 1$

or $x = e$

Again,

$$f''(x) = \frac{x^{1/x}}{x^2}\left[0 - \frac{1}{x}\right] + (1 - \log x)\frac{d}{dx}\left(\frac{x^{1/x}}{x^2}\right)$$

$$\therefore \quad f''(e) = \frac{e^{1/e}}{e^2}\left(-\frac{1}{e}\right) + 0$$

$\Rightarrow f''(e) < 0$

$\therefore$ 'f' has a maximum at x = e.

But x = e is the only extreme value.

$\therefore$ f has the greatest value at x = e

$\Rightarrow f(e) > f(\pi)$ for all $x > 0$

$\Rightarrow (e)^e > (\pi)^{1/\pi}$

$\Rightarrow e^\pi > \pi^e$

Sol.2 (A)

At point of maxima $f'(x) = 0$ and $f''(x) > 0$

$\Rightarrow f''(x) = x^2 - f^2(x) \le 0$

{Since the curve is $x^2 - y^2 = a^2 - f^2(x) \le 0$

$\therefore x_1^2 - y_1^2 < a^2 \Rightarrow$ point lies outside the hyperbola}

$\Rightarrow$ Point P(x, f(x)) lies outside $x^2 - y^2 = a^2$.

$\therefore$ Two normal can be drawn.

Sol.3 (C)

Here, $f'(x) = (b^2 - 3b + 2).(-2\sin 2x) + (b - 1)$

$f'(x)$ does not possesses stationary points

$\Rightarrow f'(x) \ne 0$

$\Rightarrow (b - 1)(b - 2)(2 - 2\sin 2x) + (b - 1) \ne 0$ for any $x \in R$

$\Rightarrow (b - 1)\{1 - 2(b - 2)\sin 2x\} \ne 0$

$$\Rightarrow \left[\frac{1}{1(b-2)}\right] > 1 \text{ and } b \ne 1$$

$$\Rightarrow -\frac{1}{2} < b - 2 < \frac{1}{2} \text{ and } b \ne 1$$

$$\Rightarrow b \in \left(\frac{3}{2}, \frac{5}{2}\right)$$

Sol.4 (D)

Here, $g(x) = f(\cot^2 x + 2\cot x + 2)$

$\Rightarrow g'(x) = f'(\cot^2 x + 2\cot x + 2).$
$\{-2\cot x\,\text{cosec}^2 x - 2\text{cosec}^2 x\}$

For g(x) to be decreasing, $g'(x) < 0$

$\Rightarrow f'\{(\cot x + 1)^2 + \}.$
$(-2\text{cosec}^2 x - 2\text{cosec}^2 x)(\cot x + 1)$, 0

$\Rightarrow f'\{(\cot x + 1)^2 = 1\}.(\cot x + 1) > 0 \ldots(1)$

{as $f''(x) > 0 \Rightarrow f'(x)$ is increasing,

then $f'\{(\cot x + 1)^2 + 1\} > f'(1)$

$$= 0 \ \forall\ x \in \left(0, \frac{3\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \pi\right)\}$$

Thus, equation (i) holds, if $\cot x + 1 > 0$

$$\Rightarrow \cot x > -1\ \forall\ x \in \left(0, \frac{3\pi}{4}\right)$$

Sol.5 (B)

Consider the function given by

$$f(x) = \int (\sin x + x\cos x)\,dx = x\sin x$$

We observe that $f(0) = f(\pi) = 0$

$\therefore$ 0 and π are two roots of (x) = 0.

Consequently, $f'(x) = 0$ i.e. $\sin x + x\cos x = 0$ has at least one root in $(0, \pi)$

Sol.6 (B)

We have $a = \frac{d^2x}{dt^2} = -9.8$. The initial conditions are $x(0) = 19.6$ and $v(0) = 0$.

So $v = \frac{dx}{dt} = -9.8t + v(0) = -9.8t$

$\therefore \quad x = -4.9t^2 + x(0) = -4.9t^2 + 19.6$

Now, the domain of the function is restricted since the ball hits the ground after a certain time. To find this time we set x = 0 and solve for t.

$0 = -4.9t^2 + 19.6 \quad \Rightarrow t = 2$

Thus $\quad x = -4.9t^2 + 19.6 \quad (0 \le t \le 2)$

Sol.7 (A)

let H be the height of the cone and α be its semi vertical angle. Suppose that x is the radius of the inscribed cylinder and h be its height $h = QL = OL - OQ = H - x\cot\alpha$

V = volume of the cylinder $= \pi x^2 (H - x\cot\alpha)$

Also $p = \frac{1}{3}\pi (H\tan\alpha)^2 \quad ...(1)$

$$\frac{dV}{dx} = \pi(2Hx - 3x^2\cot\alpha)$$

so $\frac{dV}{dx} = 0 \Leftrightarrow x = 0, x = \frac{2}{3} H\tan\alpha$,

$$\left.\frac{d^2V}{dx^2}\right|_{x=\frac{2}{3}H\tan\alpha} = 2\pi H < 0.$$

So V is maximum when $x = \frac{2}{3} H\tan\alpha$ and

$$q = V_{max} = \pi \frac{4}{9} H^2 \tan^2\alpha \frac{1}{3} H = \frac{4}{9} P.$$

[using (1)]

Hence p : q = 9 : 4

Sol.8 (D)

Any point on $y^2 = 4x$ is $(t^2, 2t)$.

Since $\frac{dy}{dx} = \frac{2}{y}$ so $\left.\frac{dy}{dx}\right|_{(t^2,2t)} = \frac{2}{2t} = \frac{1}{t}$.

Hence equation of normal at (t^2, t) is

$y - 2t = -t(x - t^2)$

This passes through (c, 0) if

$-2t = -t(c - t^2)$

$\Rightarrow \quad t = 0, t^2 = c - 2$

Thus the roots are real and distinct if $c > 2$, so $c = 3$ is the correct choice.

Sol.9 (D)

The given function can be written as

$f(x) = (a - 1)(a - 2)\cos(x/2) + (a - 1)x + \sin 1$.

It is clearly differentiable, so its critical points are given by

$f'(x) = (-1/2)(a - 1)(a - 2)\sin(x/2) + (a - 1) = 0$

If a = 1, f'(x) 0 for all x, while for values of a other than 1, f(x) will be zero if

$(a - 2)\sin(x/2) = 2$

In order not to have critical point, a must therefore satisfy $a \ne 1$ and $\left|\frac{2}{a-2}\right| > 1$

which is the same as saying $a \in (0, 1) \cup (1, 4)$

Sol.10 (B)

For a > 0, the curves, $y_1 = ax^2$ and $y_2 = \log x$ can have only one point in common if they touch each other. At the point of tangency $y'_1(x) = y'_2(x) \Rightarrow 2ax = 1/x \Rightarrow x = 1/\sqrt{2a}$ (clearly x cannot be negative).

Putting this value in $ax^2 = \log x$,

we have $(1/2) = \log(2a)^{-1/2}$

$\Rightarrow \log 2a = -1 \Rightarrow a = 1/2e$.