Dear student following is a Moderate level [O O ● O O] test paper. Score of 18 Marks in 10 Minutes would be a satisfactory performance. Questions 1-10(+3,–1) (Questions may have more than one option correct)

Direction (1 - 2) :

If $f(x + y) = f(x) + f(y)$ for all $x, y \in R$ and

$f(1) = 1$ and $g(0) = \lim_{x\to 0} \dfrac{2^{f(\tan x)} - 2^{f(\sin x)}}{x^2 f(\sin x)}$, then

Q.1 The value of f(x) is :
(A) x (B) x^2 (C) $3x$ (D) None

Q.2 The value of g(x) is :
(A) $\log_e 2$ (B) $\frac{1}{2}\log_e 2$
(C) $2\log_e 2$ (D) $\log_e\left(\frac{1}{2}\right)$

Q.3 The value of f(0), for $f(x) = \left(1 + \tan^2\sqrt{x}\right)^{1/2x}$, so that f(x) is continuous everywhere, is-
(A) e (B) 1/2 (C) $e^{1/2}$ (D) 0

Q.4 The points of discontinuity of the function $f(x) = \dfrac{1+\cos 5x}{1-\cos 4x}$ are-
(A) $x = 0$ (B) $x = \frac{\pi}{2}$
(C) $x = \pi$ (D) $x = \frac{\pi}{4}$

Q.5 Let [.] denote the greatest integer function and $f(x) = [\tan^2 x]$, then :
(A) $\lim_{x\to 0} f(x)$ does not exist
(B) f(x) is continuous at x = 0
(C) f(x) is not differentiable at x = 0
(D) $f'(0) = 1$

Q.6 The function $f(x) = [x]^2 - [x^2]$ (where [y] is the greatest integer less than or equal to y), is discontinuous at :
(A) All integers
(B) All integers except 0 and 1
(C) All integers except 0
(D) All integers except 1

Q.7 Let f be a function such that
$f(xy) = f(x).f(y),\ \forall\ y \in R$
and $f(1 + x) = 1 + x(1 + g(x))$,
where $\lim_{x\to 0} g(x) = 0$
Then the value of $\int_1^2 \frac{f(x)}{f'(x)} \cdot \frac{1}{1+x^2}\,dx$ is
(A) $\frac{1}{2}$ [log (5/2)] (B) log (5/2)
(C) 2 log (5/2) (D) log (2/5)

Q.8 The set of points where $x^2|x|$ is thrice differentiable is -
(A) R (B) R — {0}
(C) Z (D) None of these

Q.9 Let $f(x) = x^n$, n being non-negative integer. Then the value of n for which the equality $f'(a + b) = f'(a) + f'(b)$ is valid for all, a, b > 0
(A) n = 0 (B) n = 2
(C) n = 1 (D) None of these

Q.10 Let f(x) = maximum $\{4, 1 + x^2, x^2 - 1\}\ \forall\ x \in R$. Then the total number of points, where f(x) is not differentiable are
(A) 2 (B) 1 (C) 3 (D) 0

MATHEMATICS IIT JEE (JULY 1st WEEK CLASS TEST 3) (CONTINIUITY & DIFFERENTIABILITY) ANSWER KEY

Name : .... Roll No. : ....

	A	B	C	D		A	B	C	D		A	B	C	D
1	○	○	○	○	4	○	○	○	○	7	○	○	○	○
2	○	○	○	○	5	○	○	○	○	8	○	○	○	○
3	○	○	○	○	6	○	○	○	○	9	○	○	○	○
										10	○	○	○	○

ANSWER KEY

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	A	B	C	A,B,C	B	B	A	B	A,B	A

SOLUTIONS

Sol.1-2

1 - (A), 2 - (B)

We have, $f(x+y) = f(x) + f(y)$ for all $x, y \in R$

$\Rightarrow f(x) = xf(1)$ for all $x \in R$

$\Rightarrow f(x) = x$ for all $x \in R$ $\{\because f(1) = 1$, given$\}$

$$\text{Now, } g(0) = \lim_{x\to 0} \frac{2^{f(\tan x)} - 2^{f(\sin x)}}{x^2 f(\sin x)}$$

$$= \lim_{x\to 0} \frac{2^{\tan x} - 2^{\sin x}}{x^2.\sin x}$$

$$= \lim_{x\to 0} \frac{(2^{\tan x - \sin x} - 1).2^{\sin x}}{x^2 \sin x}$$

$$= \lim_{x\to 0} \frac{(2^{\tan x - \sin x} - 1)}{(\tan x - \sin x)} . \frac{\tan x - \sin x}{x^2 \sin x} . 2^{\sin x}$$

$$= \log_e 2 . \frac{1}{2} .2^0 \left\{\because \lim_{x\to 0} \frac{\tan x - \sin x}{x^2 \sin x} = \frac{1}{2}\right\}$$

$$= \frac{1}{2} \log_e 2$$

Sol.3 (C)

Put $u = \tan^2 \sqrt{x}$,

$$f(x) = (1+u)^{(1/u).(1/2)\left((\tan\sqrt{x})/\sqrt{x}\right)^2}$$

$$\Rightarrow \underset{x\to 0}{\text{Lt}} f(x) = \underset{u\to 0}{\text{Lt}} (1+u)^{1/2u} = e^{1/2}$$

Sol.4 (A, B, C)

f(x) is clearly continuous except at the points where $1 - \cos 4x = 0$. Thus $x = 0$, $x = \frac{\pi}{2}$ and $x = \pi$ are the points of discontinuity.

Sol.5 (B)

$f(x) = [\tan^2 x]$ [given]

if $-45^o < x < 45^o$

$\Rightarrow \tan(-45^o) < \tan x < \tan(45^o)$

$\Rightarrow -\tan 45^o < \tan x < \tan(45^o)$

$\Rightarrow -1 < \tan x < 1$

$\Rightarrow 0 < \tan^2 x < 1$

$\Rightarrow [\tan^2 x] = 0.$

i.e. f(x) is zero for all values of x from $x = -45^o$ to 45^o. Thus, f(x) exists when $x \to 0$ and also it is continuous at $x = 0$, f(x) is differentiable at $x = 0$ and has a value of zero.

Sol.6 (B)

All integers are critical point for greatest integer function.

So, **Case 1 : $x \in I$**

$f(x) = [x]^2 - [x^2] = x^2 - x^2$

$= 0$

Case 2 : $x \notin I$

If $0 < x < 1$, then $[x] = 0$

and $0 < x^2 < 1$, then $[x^2] = 0$

Next, if $1 < x^2 < 2$

$\Rightarrow 1 < x < \sqrt{2}$ $\Rightarrow [x] = 1$ and $[x^2] = 1$

$\therefore f(x) = [x]^2 - [x^2] = 0$ if $1 < x < \sqrt{2}$

$\therefore f(x) = 0$, if $0 \le x < \sqrt{2}$

This shows that f(x) is continuous at $x = 1$

Therefore, f(x) is discontinuous in $(-\infty, 0) \cup [\sqrt{2}, \infty)$ on many other points.

Sol.7 (A)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f\left(x\left(1+\frac{h}{x}\right)\right) - f(x)}{h}$$

$$\Rightarrow f'(x) = \lim_{h\to 0} \frac{f(x).f\left(1+\frac{h}{x}\right) - f(x)}{h}$$

[Given $f(xy) = f(x).f(y)$]

$$\Rightarrow f'(x) = \lim_{h\to 0} \frac{f(x).\left\{1+\frac{h}{x}(1+g(h/x))\right\} - f(x)}{h}$$

[Given $f(1 + x) = 1 + x(1 + g(x))$]

$$\Rightarrow f'(x) = \lim_{h\to 0} \frac{f(x)\left\{1+\frac{h}{x}(1+g(h/x))-1\right\}}{h}$$

$$\Rightarrow f'(x) = \frac{f(x)}{x} \quad .... (i) \left[\text{as } \lim_{x\to 0} g(x) = 0\right]$$

$$\therefore \int_1^2 \frac{f(x)}{f'(x)} \cdot \frac{1}{1+x^2}\,dx$$

$$= \int_1^2 \frac{x}{1+x^2}\,dx \qquad \text{[Using (i)]}$$

$$= \frac{1}{2}\,[\log|1+x^2|]_1^2$$

$$\Rightarrow \frac{1}{2}\,[\log(5/2)]$$

Sol.8 (B)

Let $f(x) = x^2\,|x|$ which could be expressed as,

$$f(x) = \begin{cases} -x^3 &, x<0 \\ 0 &, x=0 \\ x^3 &, x>0 \end{cases}$$

$$\text{This gives, } f'(x) = \begin{cases} -3x^2 &, x<0 \\ 0 &, x=0 \\ 3x^2 &, x>0 \end{cases}$$

So, $f'(x)$ exists for all real x.

$$f''(x) = \begin{cases} -6x &, x<0 \\ 0 &, x=0 \\ 6x &, x>0 \end{cases}$$

So, $f''(x)$ exists for all real x.

$$f'''(x) = \begin{cases} -6 &, x<0 \\ 0 &, x=0 \\ 6 &, x>0 \end{cases}$$

However, $f'''(0)$ does not exists since $f'''(0^-) = -6$ and $f'''(0^+) = 6$ which are not equal. Thus, the set of points where f(x) is thrice differentiable is $R - \{0\}$.

Sol.9 (A, B)

Since $f(x) = x^n$, n being non-negative integer.

Then $f'(x) = nx^{n-1}$

$f'(a) = na^{n-1}$, $f'(b) = nb^{n-1}$,

$f'(a+b) = n(a+b)^{n-1}$

Now the equality $f'(a+b) = f'(a) + f'(b)$ holds if,

$n(a+b)^{n-1} = na^{n-1} + nb^{n-1}$

or $(a+b)^{n-1} = a^{n-1} + b^{n-1}$ .... (1)

Above statement is true, only if $(n-1) = 1$

$\Rightarrow n = 2$

i.e., $(a+b)^{n-1} = a^{n-1} + b^{n-1}$ (if n = 2)

or $(a+b)^1 = a^1 = b^1$

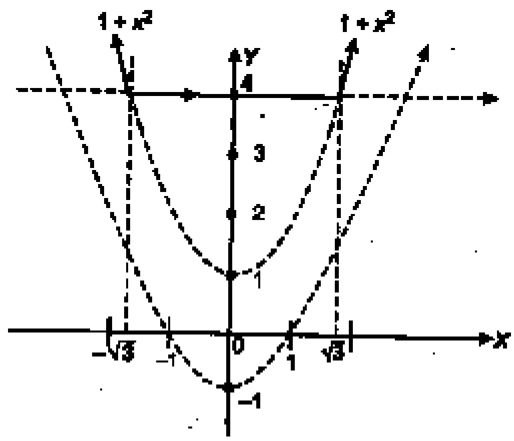
also when n = 1, 3, 4, 5, ... then,

LHS > RHS

Again when n = 0, $f'(x) = 0$ for all x

So, the equality is true for n = 0 and 2.

Sol.10 (A)



Thus, from above graph we can simply say,

f(x) is not differentiable at $x = \pm\sqrt{3}$

And it could be defined as :

$$f(x) = \begin{cases} 4 &, -\sqrt{3} \le x \le \sqrt{3} \\ x^2+1 &, x \le -\sqrt{3} \text{ or } x \ge \sqrt{3} \end{cases}.$$