Dear student following is a Moderate level [○ ○ ● ○ ○] test paper. Score of 15 Marks in 10 Minutes would be a satisfactory performance. Questions 1-9(+3,–1) (Questions may have more than one option correct)

Q.1 No. of points of discontinuity of the function $f(x) = [x] + [-x]$ is

(A) ∞ (B) 2

(C) 0 (D) 1

Q.2 Let R be a set of real nos. and $f : R \to R$ be such that for all x and y in R, $|f(x) - f(y)| \le |x - y|^3$. Then f(x) is a

(A) a const. function (B) equal to x

(C) is equal to $x - y$ (D) None of these

Q.3 Let $f(x + y) = f(x) + f(y) + 2xy - 1$ for all real x and y and f(x) be differentiable function. If $f'(0) = \cos \alpha$, then

(A) $f(x) < 0$ (B) $f(x) > 0$

(C) $f(x) \ge 0$ (D) $f(x) \le 0$

Q.4 If f(x) is a real value function not identically equal to zero. Such that $f(x + y^n) = f(x) + f(y)^n$, $x, y \in R$, n is natural no. > 1 and $f'(0) > 0$. Then value of f(5) & f'(10) are

(A) 1, 5 (B) 0, 1

(C) 5, 1 (D) 5, 0

Q.5 No. of points of discontinuity of function g(x)

$$= \lim_{n\to\infty} \frac{x^{2n} - 1}{x^{2n} + 1} \text{ are}$$

(A) 1 (B) 2

(C) zero (D) ∞

Q.6 If $f(x) = (x + [x^3 + 1])^{x^2 + \sin x}$, then indicate no. of points where it does not exist

(A) 2 (B) 0

(C) 1 (D) None

Q.7 Let f be a real function satisfying

$f(x + y + z) = f(x) f(y) f(z)$

for all real, x, y, z. If $f(2) = 4$ and $f'(0) = 3$. Then the value of f(0) and f'(2) are

(A) 1, 12 (B) 12, 1

(C) 4, 1 (D) –1, 12

Q.8 Let $f(x) = \phi(x) + \psi(x)$ and $\phi'(a)$, $\psi'(a)$ are finite and definite. Then :

(A) f(x) is continuous at $x = 0$

(B) f(x) is differentiable at $x = 0$

(C) f'(x) is continuous at $x = 0$

(D) f'(x) is differentiable at $x = 0$

Q.9 Let $f(x) = x^3 - x^2 + x + 1$ and

$$g(x) = \begin{cases} \max. f(t); & 0 \le t \le x \text{ for } 0 \le x \le 1 \\ 3 - x; & 1 < x \le 2 \end{cases}$$

then

(A) g(x) is continuous for all $x \in [0, 2]$

(B) g(x) is not differentiable at $x = 1$

(C) g(x) is differentiable at $x = 1$

(D) g(x) is discontinuous for all $x \in [0, 2]$

MATHEMATICS IIT JEE (JULY 2nd WEEK CLASS TEST 2) (CONTINIUITY & DIFFERENTIABILITY) ANSWER KEY

Name : .... Roll No. : ....

	A	B	C	D		A	B	C	D		A	B	C	D
1	⬭	⬭	⬭	⬭	4	⬭	⬭	⬭	⬭	7	⬭	⬭	⬭	⬭
2	⬭	⬭	⬭	⬭	5	⬭	⬭	⬭	⬭	8	⬭	⬭	⬭	⬭
3	⬭	⬭	⬭	⬭	6	⬭	⬭	⬭	⬭	9	⬭	⬭	⬭	⬭

ANSWER KEY

Que.	1	2	3	4	5	6	7	8	9
Ans.	A	A	B	C	B	A	A	A,B	A,B

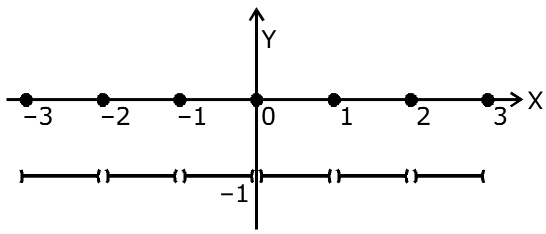
SOLUTIONS

Sol.1 (A)

$$f(x) = [x] + [-x] = \begin{cases} x - x & x \in \text{integers} \\ [x] - 1 - [x] & x \notin \text{integers} \end{cases}$$

$\therefore \quad f(x) = \begin{cases} 0 & x \in \text{integers} \\ -1 & x \notin \text{integers} \end{cases}$

Which shows the graph of f(x) as



Thus f(x) is discontinuous at $x \in$ integers.

Sol.2 (A)

We know

$$\lim_{x\to 0} |f(x)| \to 0 \Leftrightarrow f(x) \to 0 \quad ....(1)$$

and $|f(x) - f(y)| \le y\,|x - y|^3\ \forall\ x, y \in R$.

Let x be any real no. and y be choosen in the neighbourhood of x, but not equal to x.

$\therefore \quad \left|\frac{f(x)-f(y)}{x-y}\right| \le |x - y|^2$

Now taking $\lim_{y\to x}$ on both sides and using (1)

$\Rightarrow \quad \lim_{y\to x}\left|\frac{f(x)-f(y)}{x-y}\right| \le \lim_{y\to x}|x-y|^2$

$\therefore \quad |f'(x)| \le 0$

i.e. $|f'(x)| = 0$ {since absolute value could never be –ve}

or $\quad f'(x) = 0$

$\Rightarrow \quad f(x) =$ constant function.

Sol.3 (B)

We have

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x)+f(h)+2hx-1-f(x)}{h}$$

[using given definition]

$$= \lim_{h\to 0}\left\{2x + \frac{f(h)-1}{h}\right\}$$

Now substituting x = y = 0 in the given functional relation, we get,

$f(0) = f(0) + f(0) + 0 - 1$

$\Rightarrow \quad f(0) = 1$

$\therefore \quad f'(x) = 2x + \lim_{h\to 0}\frac{f(h)-f(0)}{h} = 2x + f'(0)$

$\Rightarrow \quad f'(x) = 2x + \cos\alpha$

Integrating,

$f(x) = x^2 + x\cos\alpha + C$

Here, x = 0 and f(0) = 1

$\therefore \quad 1 = C$

$\Rightarrow \quad f(x) = x^2 + x\cos\alpha + 1$

It is a quadratic in x with discriminant

$D = \cos^2\alpha - 4 < 0$

and coefficient of $x^2 = 1 > 0$

$\therefore \quad f(x) > 0 \quad \forall\ x \in R$

Sol.4 (C)

Put x = y = 0

$\Rightarrow \quad f(0) = 0$

Now $\lim_{y\to 0}\frac{f(x+y^n)-f(x)}{y^n} = \lim_{y\to 0}\frac{(f(y))^n}{y^n}$

$\Rightarrow \quad f'(x) = \left(\lim_{y\to 0}\frac{f(y)}{y}\right)^n$

$\Rightarrow \quad f'(x) = \left(\lim_{y\to 0}\frac{f(y+0)-f(0)}{y}\right)^n = (f'(0))^n$

$\Rightarrow \quad f'(x) = A \qquad \{A = f'(0)\}$

$\Rightarrow \quad f(x) = Ax + B$

$\{f'(0) = 1 = A \text{ and } x = 0 \Rightarrow f(0) = 0\}$

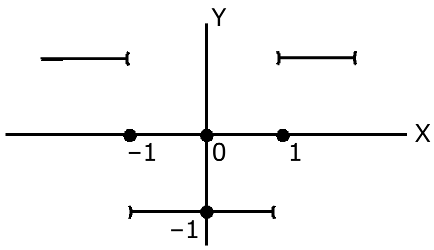
$\therefore \quad f(x) = x$

Hence f(5) = 5 and f'(10) = 1.

Sol.5 (B)

$$g(x) = \lim_{n\to\infty} \frac{x^{2n}-1}{x^{2n}+1} = \begin{cases} -1 & |x| < 1 \\ 0 & |x| = 1 \\ 1 & |x| > 1 \end{cases}$$

i.e.



Thus, shows g(x) is discontinuous at x = ±1.

Sol.6 (A)

Here, $f(x) = (x + [x^3 + 1])^{x^2 + \sin x}$

$$f(x) = \begin{cases} (x+2)^{x^2+\sin x}, & 1 < x < 2^{1/3} \\ (x+3)^{x^2+\sin x}, & 2^{1/3} < x < 3^{1/3} \\ (x+4)^{x^2+\sin x}, & 3^{1/3} < x < \frac{3}{2} \end{cases}$$

as $[x^3 + 1] = \begin{cases} 2, & 1 < x < 2^{1/3} \\ 3, & 2^{1/3} < x < 3^{1/3} \\ 4, & 3^{1/3} < x < \frac{3}{2} \end{cases}$

which shows f(x) is discontinuous at $x = 2^{1/3}$ and $3^{1/3}$ and so not differentiable at $x = 2^{1/3}$ and $3^{1/3}$.

Sol.7 (A)

Here, f(x + y + z) = f(x) f(y) f(z) ....(i)

for all x, y, z ∈ R

Put x = y = z = 0

$f(0) = (f(0))^3$

⇒ f(0) = 0, ±1 ....(ii)

Putting y = z = −1 in (i), we get

$f(x - 2) = f(x)\{f(-1)\}^2$

⇒ $f(0) = f(2)\{f(-1)\}^2$ for all x ∈ R

⇒ $f(0) = 4\{f(-1)\}^2$

⇒ f(0) > 0 ....(iii)

∴ from (ii) and (iii),

f(0) = 1 ....(iv)

Now putting y = 2 and z = 0 in (i), we get

f(x + 2) = f(x) f(2) f(0)

f(x + 2) = 4f(x)

f'(x + 2) = 4f'(x), putting x = 2

∴ f'(4) = 4, f'(2) = 12

Thus, f(0) = 1 and f'(2) = 12

Sol.8 (A, B)

We know that the sum of two continuous (differentiable) functions is continuous (differentiable).

∴ f(x) is continuous and differentiable at x = a.

Sol.9 (A, B)

Here $f(x) = x^3 - x^2 + x + 1$

⇒ $f'(x) = 3x^2 - 2x + 1$ which is strictly increasing in (0, 2).

∴ $g(x) = \begin{cases} f(x); & 0 \le x \le 1 \\ 3 - x; & 1 < x \le 2 \end{cases}$

[as f(x) is increasing so, f(x) is maximum when 0 ≤ t ≤ x]

So, $g(x) = \begin{cases} x^3 - x^2 + x + 1; & 0 \le x \le 1 \\ 3 - x; & 1 < x \le 2 \end{cases}$

also, $g'(x) = \begin{cases} 3x^2 - 2x + 1; & 0 \le x \le 1 \\ -1; & 1 < x \le 2 \end{cases}$

Which clearly shows g(x) is continuous for all x ∈ [0, 2] but g(x) is not differentiable at x = 1.